Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 3, 2003*
Revised: *June 23, 2003*
Accepted: *June 26, 2003*

Two-dimensional non-commutative Yang-Mills theory: coherent effects in open Wilson line correlators

Antonio Bassetto,* Giancarlo De Pol and Federica Vian

Dipartimento di Fisica G.Galilei
Via Marzolo 8, 35131 Padova, Italy, and
INFN, Sezione di Padova, Italy
E-mail: bassetto@pd.infn.it, depol@pd.infn.it, vian@pd.infn.it

Abstract: A perturbative calculation of the correlator of three parallel open Wilson lines is performed for the U(N) theory in two non-commutative space-time dimensions. In the large-N planar limit, the perturbative series is fully resummed and asymptotically leads to an exponential increase of the correlator with the lengths of the lines, in spite of an interference effect between lines with the same orientation. This result generalizes a similar increase occurring in the two-line correlator and is likely to persist when more lines are considered provided they share the same direction.

Keywords: Field Theories in Lower Dimensions, 1/N Expansion, Non-Commutative Geometry.

*Partially supported by the European Community network HPRN-CT-2000-00149.

Contents

1. Introduction

Interesting dynamical information on gauge theories defined on noncommutative space-time are provided by correlation functions of gauge invariant operators [1].

Noncommutativity of D-dimensional Minkowski space-time is encoded in a real anti-symmetric matrix $\theta^{\mu\nu}$:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \qquad \mu, \nu = 0, \ldots, D-1 \tag{1.1}$$

and a $\star$-product of two fields $\phi_1(x)$ and $\phi_2(x)$ can be defined by means of Weyl symbols

$$\phi_1 \star \phi_2(x) = \int \frac{d^D p\, d^D q}{(2\pi)^{2D}} \exp\left[-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu\right] \exp(ipx) \tilde{\phi}_1(p-q) \tilde{\phi}_2(q)\,. \tag{1.2}$$

Then noncommutative gauge theories (NCGT)[1] are most easily formulated by replacing the usual multiplication of fields in the lagrangian with the Moyal $\star$-product. The resulting action makes them obviously non-local. As a consequence, gauge invariance in this case (star-gauge invariance) entails an integration over space-time variables and the possibility of having *local* probes is lost.

There is however a remarkable recipe in NCGT which turns local operators into gauge invariant observables carrying a non-vanishing momentum [3]. *Open* Wilson lines $W(p)$ with momentum p_μ can be considered which are gauge invariant provided their length l^ν is related to the momentum as follows

$$l^\nu = p_\mu \theta^{\mu\nu}\,. \tag{1.3}$$

[1]Reviews on NCGT may be found for instance in [2].

Then, averaging any usual *local* gauge invariant operator with respect to space-time and group variables with a weight given by an open Wilson line provides indeed an (overcomplete) set of dynamical observables. The Wilson line itself provides the simplest example of such an average, the *local* operator being just unity in this case [3, 4].

In [4] a perturbative estimate in the 't Hooft limit of open Wilson line correlators was carried out in four dimensions for a $\mathcal{N} = 4$ supersymmetric U(N) NCGT, with θ of a "magnetic" type. In four dimensions ladder diagrams dominate. An all order resummation was compared with a dual supergravity result and a good agreement was found. The correlator of *two* (necessarily parallel) lines did exhibit an exponential increase with the length of the lines whereas the *normalized* correlators of multiple Wilson lines were found to be exponentially decreasing. This result was ascribed to the lack of parallelism (and thereby of coherence) which generally occurs when several lines are considered.

In two dimensions noncommutativity necessarily involves the time variable, but the Lorentz symmetry is not violated owing to the tensorial character of $\theta^{\mu\nu}$. In the light-cone gauge, the perturbative calculation is greatly simplified, thanks to the decoupling of Faddeev-Popov ghosts and to the vanishing of the vector vertices. When considering the correlator of two open Wilson lines, it turns out that in all diagrams which are leading in the large-N limit, the contributions of θ-dependent phases cancel after volume integration [5]. This is far from trivial because cyclic permutations inside the trace associated to each Wilson line do not entail any change in the related colour factors; as a consequence leading diagrams as far as colour is concerned are not necessarily of a ladder type. Nevertheless it turns out they contribute in the limit of large length of the lines on an equal footing with the simpler ladder graphs; this is the reason why such diagrams were still called "planar" in [5]. There a detailed perturbative calculation of the correlator of two Wilson lines was performed in two dimensions in the 't Hooft limit $N \to \infty$, g^2N fixed. After a full resummation of the perturbative series, an exponential increase of the correlator with respect to the length of the lines was found, in agreement with the analogous behaviour occurring in four dimensions. Such a result was also confirmed by a non perturbative calculation on a bidimensional torus using Morita equivalence, followed by decompaction, in a suitable region of the parameters involved, corresponding to the planar regime as defined in [6].

We stress that in order to reach this goal, the large-N limit was essential. Indeed, in the perturbative calculation a plethora of diagrams, subleading as far as colour is concerned, was disregarded; on the non-perturbative side, a particular saddle-point approximation, describing the mentioned planar phase of the theory, was crucial. Such an approximation requires the limit $N \to \infty$, with N larger than the winding number associated to the momentum carried by the lines and, in addition, it holds within a particular region of the parameters involved [5]. As a consequence the conclusions in [7] do not apply to that treatment; indeed the harmonic analysis performed there on the noncommutative torus is valid for finite values of N and is not suitable to infer predictions on the large-N planar regime. In passing we notice that the planar regime was also considered in the four-dimensional case [4].

It is the purpose of this paper to generalize the perturbative calculation in [5] to correlators of three *parallel* Wilson lines in the same "planar" context. This is not an academic task, since multiple line correlators in a generic configuration are not expected

JHEP06(2003)051

to increase with the length of the lines on the basis of the estimate in [4]. Remarkably, we find instead they keep increasing, when the lines are parallel, at the same rate as the two-line correlator, in spite of the fact that exchanges between lines with the same orientation generate an interference effect. This is indeed the case, but is overwhelmed by the coherent increase due to parallelism of lines with opposite orientation.

As a consequence, if a decrease would occur, it should be ascribed only to the lack of parallelism; unfortunately, we are unable to extend our calculation to this case and therefore have no new prediction concerning non-parallel lines.

The outline of the paper is as follows. In section 2 we define the three-line correlator and introduce the basic quantities and notations we will use throughout the paper. Section 3 is concerned with the perturbative calculation at any order of a generic configuration of the diagrams with the only restriction to *parallel* Wilson lines. In section 4 the perturbative series are concretely summed obtaining an asymptotic expression for the increase of the correlator with the line lengths. Final considerations and comments concerning possible future developments are considered in the conclusions, while technical details are deferred to the appendices A and B.

2. The three-line correlator

The classical action of the U(N) Yang-Mills theory on a noncommutative two-dimensional space is

$$S = -\frac{1}{2}\int d^2x \ \mathrm{Tr}\, F_{\mu\nu} \star F^{\mu\nu}\,, \tag{2.1}$$

where the field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu \star A_\nu - A_\nu \star A_\mu) \tag{2.2}$$

and $A_\mu = A^a_\mu T^a$ is a $N \times N$ matrix, with T^a normalized as follows: $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$, a, b denoting U(N) indices.

The action eq. (2.1) is invariant under infinitesimal U(N) noncommutative gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu \lambda - ig(A_\mu \star \lambda - \lambda \star A_\mu)\,. \tag{2.3}$$

As noticed in [4], under this transformation the operator $\mathrm{Tr}\, F^2(x)$ is not left invariant

$$\mathrm{Tr}\, F^2(x) \longrightarrow \mathrm{Tr}\ \mathrm{U}(x) \star F^2(x) \star U^\dagger(x)\,, \tag{2.4}$$

with $\mathrm{U}(x) = \exp_*(ig\lambda(x))$. To recover a gauge invariant operator, one has to integrate over all space, since $\star$-products inside integrals can be cyclically permuted.

A Wilson line of length l can be defined by means of the Moyal product as [1]

$$\Omega_\star[x, C] = P_\star \exp\left(ig\int_0^l A_\mu(x + \zeta(\sigma))\, d\zeta^\mu(\sigma)\right), \tag{2.5}$$

where C is the curve parameterized by $\zeta(\sigma)$, with $0 \leq \sigma \leq 1$, $\zeta(0) = 0$, $\zeta(1) = l$, and $P_\star$ denotes noncommutative path ordering along $\zeta(\sigma)$ from right to left with respect to increasing σ of $\star$-products of functions. The Wilson line is not invariant under a gauge transformation

$$\Omega_\star[x, C] \longrightarrow \mathrm{U}(x) \star \Omega_\star[x, C] \star U^\dagger(x + l)\,. \tag{2.6}$$

In order to recover gauge invariance one should perform a complete trace operation, which, in a noncommutative context, entails also integration over coordinates.

The following operator

$$W(p, C) = \int d^2x \; \mathrm{Tr}\, \Omega_\star[x, C] \star e^{ipx}\,, \tag{2.7}$$

turns out to be invariant provided C satisfies the condition

$$l^\nu = p_\mu \theta^{\mu\nu} \tag{2.8}$$

(the Wilson line extends in the direction transverse to the momentum). The particular case $p_\mu = 0$ corresponds to a closed loop.

For simplicity in the following only straight lines will be considered. Then one can easily realize that any local operator $\mathcal{O}(x)$ in ordinary gauge theories admits a noncommutative generalization

$$\tilde{\mathcal{O}}(p) = \mathrm{Tr} \int d^2x\, \mathcal{O}(x) \star \Omega_\star[x, C] \star e^{ipx}\,, \tag{2.9}$$

each of the $\tilde{\mathcal{O}}(p)$'s being a genuinely different operator at different momentum.

Remarkably, owing to eq. (2.8), at large values of $|p|$, gauge invariance requires that the length of the Wilson line becomes large. This fact can be interpreted as a manifestation of the UV-IR mixing phenomenon.

The authors of [5] studied the two-point function $\langle W(p, C) W^\dagger(p, C')\rangle$ in two space-time dimensions, where $W(p, C)$ has been defined via eqs. (2.5) and (2.7). It represents the correlation function of two straight (anti-)parallel Wilson lines of equal length, each carrying a transverse momentum p.

The theory defined through eq. (2.1) was quantized in the light-cone gauge $A_- = 0$ at *equal times*, the free propagator having the following causal expression (WML prescription)

$$D_{++}^{WML}(x) = \frac{1}{2\pi} \frac{x^-}{-x^+ + i\epsilon x^-}\,, \tag{2.10}$$

first proposed by T.T. Wu [8]. In turn this propagator is nothing but the restriction in two dimensions of the expression proposed by S. Mandelstam and G. Leibbrandt [9] in four dimensions[2] and derived by means of a canonical quantization in [11].

[2]In dimensions higher than two, where physical degrees of freedom are switched on (transverse "gluons"), this causal prescription is mandatory [10].

This form of the propagator allows a smooth transition to an euclidean formulation: $x_0 \to ix_2, x_1 \to x_1$. The propagator above becomes

$$D_E(x) = \frac{1}{2\pi}\frac{x_1 + ix_2}{x_1 - ix_2} = -\frac{1}{2\pi^2}\int d^2k e^{ikx}\frac{1}{(k_1 - ik_2)^2}\,. \tag{2.11}$$

In the sequel momentum integrals will always be performed by means of a "symmetric integration" [8], namely by an angular average around the pole. We shall also use the light-cone notation $k_- \equiv -(k_1 - ik_2)$.

The light-cone gauge gives rise to other important features like the decoupling of Faddeev-Popov ghosts, which occurs also in the noncommutative case [12], and the absence of the triple gluon vertex in two dimensions. Consequently the computation of the Wilson line correlator is enormously simplified.

The main result found in [5] is the exponential increase of the two-line correlator $\langle W(p, C)W^\dagger(p, C')\rangle$ with respect to the (equal) length of the lines C and C'.

We now consider the three-line correlator $\langle W(p_1, C)W^\dagger(p_2, C')W(p_3, C'')\rangle$; since we choose the lines to be parallel, momentum conservation

$$p_1 + p_2 + p_3 = 0$$

implies the relation

$$l_1 + l_3 = l_2 \tag{2.12}$$

among the three lengths.

With no loss of generality we choose the paths stretching along x^0, so that p_i points in the spatial direction. The parallel paths C' and C'' may be chosen at a distance Δ' and Δ'' from C, respectively. However these distances amount only to the appearance of two irrelevant phase factors (see [5]). As a consequence it is not restrictive to imagine the three lines superimposed and the dependence on C, C' and C'' will be understood in the sequel.

The variables ζ, ξ and η, parameterizing the lines C, C' and C'', respectively, can be conveniently rescaled as $\zeta = \sigma l_1$, $\xi = \sigma' l_2$ and $\eta = \sigma'' l_3$, with $0 \le \sigma, \sigma', \sigma'' \le 1$ and $l_i^0 = |\theta p_i|, i = 1, ., 3$, according to eq. (1.3).

We begin by expanding the line operators

$$\begin{aligned}
W(p_1) &= \sum_{m=0}^{\infty}(igl_1)^m \int d^2x \int_{\sigma_m > \ldots > \sigma_1} [d\sigma]\ \mathrm{Tr}\, A(x+\zeta_1) \star \ldots \star A(x+\zeta_m) \star e^{ip_1 x} \\
W^\dagger(p_2) &= \sum_{m=0}^{\infty}(-igl_2)^m \int d^2x \int_{\sigma'_m > \ldots > \sigma'_1} [d\sigma']\ \mathrm{Tr}\, A(x+\xi_m) \star \ldots \star A(x+\xi_1) \star e^{-ip_2 x} \\
W(p_3) &= \sum_{m=0}^{\infty}(igl_3)^m \int d^2x \int_{\sigma''_m > \ldots > \sigma''_1} [d\sigma'']\ \mathrm{Tr}\, A(x+\eta_1) \star \ldots \star A(x+\eta_m) \star e^{ip_3 x}\,.
\end{aligned} \tag{2.13}$$

The perturbative calculation now continues with performing suitable contractions among the operators in these equations, giving rise only to products of propagators since no internal vertices are present in such a theory. At this stage it is essential to consider the large-N limit which allows to disregard a large part of (*a priori* possible) configurations; we explicitly assume that non-leading configurations, as far as colour is concerned, can be consistently dropped.

One should keep in mind that cyclic permutations leave traces invariant and therefore several leading equivalent configurations are to be taken into account. Moreover a given propagator may either connect two different lines or start and end on the same line. This last situation will be treated separately. Finally, disconnected diagrams will not be considered; as a matter of fact they contribute only for particular values of the line momenta and trivially factorize in the product of lower order correlators. Still, connected diagrams are colour subleading with respect to disconnected ones as it will become apparent shortly. As a consequence, when normalized to two-line correlators, they vanish as $1/N$ in the 't Hooft limit. Nevertheless, if instead the coupling g is kept fixed and N is sent to ∞, the normalized three-line correlator becomes sizeable, as will be cleared in the sequel.

3. The perturbative calculation

Now we start performing contractions between operators giving rise to propagators attached to the three open lines. For the time being we postpone the discussion of diagrams with propagators beginning and ending on the same line; we will comment upon them later on.

First of all we have to single out configurations which are leading as far as colour is concerned. We denote with n_{ij} the number of propagators exchanged between the line i and the line j. If we introduce the shorthand notation

$$(a_1 \dots a_k) \equiv Tr(T^{a_1} \dots T^{a_k}) \,,$$

we can easily realize that the following cyclic pattern

$$(a_1 \dots a_{n_{13}} b_1 \dots b_{n_{12}})(b_{n_{12}} \dots b_1 c_1 \dots c_{n_{23}})(c_{n_{23}} \dots c_1 a_{n_{13}} \dots a_1) = \frac{1}{N} \cdot \left(\frac{N}{2}\right)^n \,, \qquad n \equiv n_{12} + n_{23} + n_{13} \,, \tag{3.1}$$

is colour leading.

In order to derive the equation above (and suitable generalizations), one has to perform the operations of joining and splitting different strings of colour matrices. The basic relation to be used, in the case of U(N), reads

$$T^a_{ij} T^a_{kl} = \frac{1}{2} \delta_{il} \delta_{jk} \,, \tag{3.2}$$

where the index a is to be summed over. Then, for instance, one gets

$$\begin{aligned}(a_1 \dots a_{n_{13}} b_1 \dots b_{n_{12}})(b_{n_{12}} \dots b_1 c_1 \dots c_{n_{23}})(c_{n_{23}} \dots c_1 a_{n_{13}} \dots a_1) = \\ = \frac{1}{4}(a_1 \dots a_{n_{13}} b_1 \dots b_{n_{12}-1} b_{n_{12}-1} \dots b_1 c_1 \dots c_{n_{23}-1} c_{n_{23}-1} \dots c_1 a_{n_{13}} \dots a_1)\end{aligned} \tag{3.3}$$

and the result eq. (3.1) easily follows.

These cyclic configurations are to be summed and we have now to consider the geometrical structure they entail.

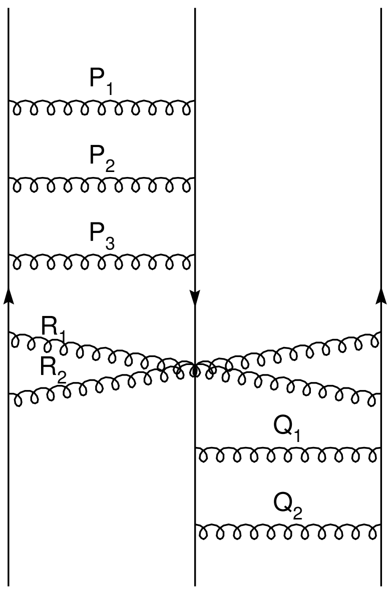

Figure 1: A non-trivial leading diagram in the large-N limit.

Starting from eq. (2.13), we contract the A's in such a way that the resulting diagram with n_{12}, n_{13}, n_{23} propagators is of leading order in N, corresponding to the colour pattern in eq. (3.1). After analytic continuation to euclidean variables, using eq. (2.11), we get at a fixed perturbative order $n = n_{12} + n_{13} + n_{23}$

$$\begin{aligned} I_{n_{12},n_{13},n_{23}} = {} & (-1)^{n_{12}+n_{23}} N^{n-1} \left(\frac{g^2}{4\pi^2}\right)^n \int \prod_{i=1}^{n_{12}} d\zeta_i \, d\xi_i \prod_{j=1}^{n_{23}} d\xi'_j \, d\eta_j \prod_{k=1}^{n_{13}} d\zeta'_k \, d\eta'_k \times \\ & \times \int \prod_{i=1}^{n_{12}} e^{-iP_i \cdot (\xi_i - \zeta_i)} \frac{d^2 P_i}{P_{i-}^2} \prod_{j=1}^{n_{23}} e^{-iQ_j \cdot (\eta_j - \xi'_j)} \frac{d^2 Q_j}{Q_{j-}^2} \times \\ & \times \prod_{k=1}^{n_{13}} e^{-iR_k \cdot (\eta'_k - \zeta'_k)} \frac{d^2 R_k}{R_{k-}^2} \, e^{i\mathcal{P}(P_i, Q_j, R_k; \theta, \mathbf{p})} \delta^{(2)} \left(\sum_i P_i + \sum_k R_k - p_1\right) \times \\ & \times \delta^{(2)} \left(\sum_j Q_j - \sum_i P_i - p_2\right) \delta^{(2)} \left(\sum_k R_k + \sum_j Q_j + p_3\right), \end{aligned} \tag{3.4}$$

where $\mathbf{p} = \{p_1, p_2, p_3\}$, the integrations over the line variables are ordered according to the colour pattern in eq. (3.1) and the Moyal phase $\mathcal{P}(P_i, Q_j, R_k; \theta, \mathbf{p})$ is a linear function of P_i, Q_j and R_k, depending on the topology.

A typical example is explicitly provided in figure 1. The corresponding analytical expression is

$$I_{3,2,2} = -N^6 \left(\frac{g^2}{4\pi^2}\right)^7 \int \prod_{i=1}^{3} d\zeta_i \, d\xi_i \prod_{j=1}^{2} d\xi'_j \, d\eta_j \prod_{k=1}^{2} d\zeta'_k \, d\eta'_k \int \prod_{i=1}^{3} e^{-iP_i \cdot (\xi_i - \zeta_i)} \frac{d^2 P_i}{P_{i-}^2} \times$$

$$\times \prod_{j=1}^{2} e^{-iQ_j\cdot(\eta_j-\xi'_j)} \frac{d^2Q_j}{Q^2_{j-}} \prod_{k=1}^{2} e^{-iR_k\cdot(\eta'_k-\zeta'_k)} \frac{d^2R_k}{R^2_{k-}} e^{-\frac{i}{2}[p_3\theta(P_1+P_2+P_3+R_1+R_2)]} \times$$

$$\times \delta^{(2)}\left(\sum_i P_i + \sum_k R_k - p_1\right) \delta^{(2)}\left(\sum_j Q_j - \sum_i P_i - p_2\right) \times$$

$$\times \delta^{(2)}\left(\sum_k R_k + \sum_j Q_j + p_3\right). \tag{3.5}$$

Following a simple but rather lengthy procedure, which is fully described in appendix A, we can rewrite eq. (3.4) in a form in which factorization of propagators in coordinate variables is manifest

$$\begin{aligned} I_{n_{12},n_{13},n_{23}} = {} & \frac{(-1)^{n_{13}}}{(2\pi)^6 N}\left(\frac{g^2N}{4\pi}\right)^n \delta^{(2)}(p_1+p_2+p_3) \int \prod_{i=1}^{n_{12}} d\zeta_i\, d\xi_i \prod_{j=1}^{n_{23}} d\xi'_j\, d\eta_j \times \\ & \times \prod_{k=1}^{n_{13}} d\zeta'_k\, d\eta'_k \int d^2x\, d^2y\, d^2z \prod_{i=1}^{n_{12}} \frac{x-a_i}{(x-a_i)^*} \prod_{j=1}^{n_{23}} \frac{y-b_j}{(y-b_j)^*} \prod_{k=1}^{n_{13}} \frac{z-c_k}{(z-c_k)^*} \times \\ & \times \int d^2P\, e^{iP\cdot(x+y+z)} e^{-\frac{i}{3}[(p_1-p_2)\cdot x+(p_2-p_3)\cdot y+(p_3-p_1)\cdot z]}, \end{aligned} \tag{3.6}$$

where $a_i = \xi_i - \zeta_i + A_i(\theta\mathbf{p})$, $b_j = \eta_j - \xi'_j + B_j(\theta\mathbf{p})$, $c_k = \eta'_k - \zeta'_k + C_k(\theta\mathbf{p})$ and A_i, B_j, C_k are linear combinations of $\{\theta p_l\}_{l=1,2,3}$, obeying $\sum_i P_iA_i + \sum_j Q_jB_j + \sum_k R_kC_k = -\mathcal{P}(P_i, Q_j, R_k; \theta, \mathbf{p})$.

Once we have recognized that the structures in eq. (3.1) allow factorization of propagators in coordinate variables, a little thought is enough to conclude that cyclic permutations on a line do not alter such a pattern. As a matter of fact any open line with n attached propagators can be considered as a diagram with $n+1$ vectors, the extra vector $p = \sum p_j$ representing the momentum balance. We know that the Moyal phase associated to *planar* diagrams only depends on external momenta and is cyclically invariant thanks to total momentum conservation [13]. One can wonder what happens if only the n vectors p_j are cyclically permuted. Since the phase is quadratic in the momenta and its variation must vanish at $p = 0$, it is easy to conclude that its variation is at most *linear* with respect to p_j. As a consequence the factorization property we mentioned is unaffected and all the configurations obtained from cyclic permutations over each line are to be considered on an equal footing, counted and summed as well. We stress that, remarkably, factorization of propagators in coordinate variables occurs just in those diagrams which are dominant at large-N. This feature in turn makes θ-dependence trivial, as it will be shortly cleared, since θ intervenes just through the length of the lines.

We find this factorization very peculiar; it is indeed characteristic of two-dimensional theories and is not fulfilled, for instance, in the four-dimensional case where planarity just means *ladder* diagrams [4].

Coming back to eq. (3.6), integration over coordinates can be performed (see again appendix A). Symmetric integration provides the natural regularization. For instance, we have

$$\int d^2x \prod_{i=1}^{n_{12}} \frac{x-a_i}{(x-a_i)^*}\, e^{i(P-\tilde{p}_1)\cdot x} = \sum_{m=1}^{n_{12}} \frac{\pi (2i)^{m+1} m!}{(P-\tilde{p}_1)_-^{m+1}} \sum_{i=1}^{n_{12}} e^{i(P-\tilde{p}_1)\cdot a_i}\, V(m,i,\mathbf{a})\,, \tag{3.7}$$

where

$$V(m,i,\mathbf{a}) = \sum_{j_1<j_2<\ldots<j_{n_{12}-m}} \frac{(a_i-a_{j_1})\ldots(a_i-a_{j_{n_{12}-m}})}{\prod_{j\neq i}(a_i-a_j)^*}\,, \tag{3.8}$$

having defined $\mathbf{a}=\{a_1,\ldots,a_{n_{12}}\}$, $\tilde{p}_1=\frac{p_1-p_2}{3}$, and similarly, for integration over y and z, $\tilde{p}_2=\frac{p_2-p_3}{3}$, $\tilde{p}_3=\frac{p_3-p_1}{3}$. Eventually, we have to perform the integration over P, following appendix A. Collecting all the terms depending on P in eq. (3.6), we get

$$\begin{aligned}
&\int d^2P\, \frac{e^{iP\cdot(a_i+b_j+c_k)}}{(P-\tilde{p}_1)_-^{m_1+1}(P-\tilde{p}_2)_-^{m_2+1}(P-\tilde{p}_3)_-^{m_3+1}} \equiv \mathcal{K}(m_1,m_2,m_3;\tilde{p}_1,\tilde{p}_2,\tilde{p}_3;a_i+b_j+c_k)\\
&\quad = 4\pi \sum_{l_2,l_3=0}^{m_1} \binom{m_2+l_2}{l_2} \frac{(-1)^{l_2}}{(\tilde{p}_1-\tilde{p}_2)_-^{m_2+l_2+1}} \binom{m_3+l_3}{l_3} \frac{(-1)^{l_3}}{(\tilde{p}_1-\tilde{p}_3)_-^{m_3+l_3+1}} \times\\
&\qquad \times \frac{\left(\frac{i}{2}\right)^{m_1-l_2-l_3+1}}{(m_1-l_2-l_3)!} \frac{(a_i+b_j+c_k)^{m_1-l_2-l_3}}{(a_i+b_j+c_k)^*} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\,.
\end{aligned} \tag{3.9}$$

Exploiting eq. (3.7), we have

$$\begin{aligned}
I_{n_{12},n_{13},n_{23}} &= \frac{(-1)^{n_{13}}}{(4\pi)^3 N}\left(\frac{g^2N}{4\pi}\right)^n \delta^{(2)}(p_1+p_2+p_3) \int \prod_{i=1}^{n_{12}} d\zeta_i\, d\xi_i \prod_{j=1}^{n_{23}} d\xi'_j\, d\eta_j\\
&\quad \times \prod_{k=1}^{n_{13}} d\zeta'_k\, d\eta'_k \sum_{m_1=1}^{n_{12}} \sum_{m_2=1}^{n_{23}} \sum_{m_3=1}^{n_{13}} (2i)^{m_1+m_2+m_3+3} m_1! m_2! m_3! \times\\
&\quad \times \sum_{i=1}^{n_{12}} \sum_{j=1}^{n_{23}} \sum_{k=1}^{n_{13}} V(m_1,i,\mathbf{a}) V(m_2,j,\mathbf{b})\, V(m_3,k,\mathbf{c}) \times\\
&\quad \times \mathcal{K}(m_1,m_2,m_3;\tilde{p}_1,\tilde{p}_2,\tilde{p}_3;a_i+b_j+c_k)\,.
\end{aligned} \tag{3.10}$$

Finally, we need to carry out some algebra. We will show in appendix A that the only contributions to the summations above are those, referring to eq. (3.9), obeying the conditions $m_r+l_r+1=2$, $r=2,3$ for the first term of the r.h.s., and analogous ones for the other two terms. One can see that the terms surviving are independent of the variables a_i, b_j and c_k. It follows that integration over geometric variables in eq. (3.10) is trivial. Just to fix ideas, integration over ζ_i, ζ'_k, produces a factor $l_1^{n_{12}+n_{13}}/(n_{12}+n_{13})!$, and so on. Thus, we end up with the following expression for eq. (3.10)[3]

$$I_{n_{12},n_{13},n_{23}} = \frac{1}{\pi^2 N}\, \delta^{(2)}(p_1+p_2+p_3) \left(\frac{g^2Nl_1l_2}{4\pi}\right)^{n_{12}} \left(-\frac{g^2Nl_1l_3}{4\pi}\right)^{n_{13}} \left(\frac{g^2Nl_2l_3}{4\pi}\right)^{n_{23}} \times$$

[3] The momentum dependence looks particularly simple thanks to our choice of coordinates in which momenta are taken along the x-direction. A more general choice would entail a phase in intermediate steps which however cancels in the final result (see [5]).

$$\times \frac{1}{(n_{12}+n_{13})!(n_{13}+n_{23})!(n_{12}+n_{23})!} \times$$
$$\times \left(\frac{n_{13}n_{23}}{|p_1|^2|p_2|^2} + \frac{n_{12}n_{23}}{|p_1|^2|p_3|^2} + \frac{n_{12}n_{13}}{|p_2|^2|p_3|^2} \right). \tag{3.11}$$

A remarkable property of eq. (3.11) is that it displays a strikingly simple dependence on the topology of the graph, namely on the integers n_{ij}.

In order to resum leading contributions in N, we just need to know the number of different configurations with fixed n_{ij}. We have already emphasized that cyclic permutations inside any trace of eq. (3.1) do not alter the power of N, so that we can conclude that the multiplicity of planar graphs amounts to $(n_{12}+n_{13})(n_{13}+n_{23})(n_{12}+n_{23})$. At this stage the full contribution to the correlator $\langle W(p_1)W^\dagger(p_2)W(p_3)\rangle$ can be readily written

$$\begin{aligned}\langle W(p_1)W^\dagger(p_2)W(p_3)\rangle = & \sum_{n_{12},n_{23},n_{13}=1} \frac{(-1)^{n_{13}}}{\pi^2 N}\,\delta^{(2)}(p_1+p_2+p_3)\times \\ & \times \frac{\tau_1^{n_{12}+n_{13}}\,\tau_2^{n_{12}+n_{23}}\,\tau_3^{n_{13}+n_{23}}}{(n_{12}+n_{13}-1)!(n_{13}+n_{23}-1)!(n_{12}+n_{23}-1)!} \times \\ & \times \left(\frac{n_{13}n_{23}}{|p_1|^2|p_2|^2} + \frac{n_{12}n_{23}}{|p_1|^2|p_3|^2} + \frac{n_{12}n_{13}}{|p_2|^2|p_3|^2} \right), \end{aligned} \tag{3.12}$$

where $\tau_i = \sqrt{\frac{g^2 N l_i^2}{4\pi}}$.

Finally, we have to take into account the effect of diagrams with "bubbles", namely with propagators beginning and ending on the same line. Again, only leading configurations with respect to N are to be considered. The counting of such configurations has already been performed in [5]. The number of ways in which we can form q pairs on a line while keeping the leading power in N turns out to be

$$S_q = \frac{2^{2q}}{(q+1)!}\frac{\Gamma(q+\frac{1}{2})}{\Gamma(\frac{1}{2})}. \tag{3.13}$$

On the other hand, the geometric factor associated with propagators starting and ending on the same line factorizes and amounts to $(-N/(4\pi))^q$ [5].

Carefully inserting these factors, taking into account the new multiplicity of planar diagrams and then summing over the number of pairs, we are led to replace eq. (3.12) with the following three expressions, according to the number of lines which are affected.

In the case of bubbles on all the three lines we get

$$\begin{aligned}\langle W(p_1)W^\dagger(p_2)W(p_3)\rangle_{\mathcal{PPP}} = & \sum_{n_{12},n_{23},n_{13}=1}\ \sum_{q_1,q_2,q_3=1} \frac{(-1)^{n_{13}}}{\pi^2 N}\,\delta^{(2)}(p_1+p_2+p_3)\times \\ \times(-1)^{q_1+q_2+q_3} & \frac{\tau_1^{n_{12}+n_{13}+2q_1}\,\tau_2^{n_{12}+n_{23}+2q_2}\,\tau_3^{n_{13}+n_{23}+2q_3}}{(n_{12}+n_{13}+2q_1-1)!(n_{13}+n_{23}+2q_2-1)!(n_{12}+n_{23}+2q_3-1)!} \\ \times S_{q_1}S_{q_2}S_{q_3}(n_{12}+n_{13}) & (n_{12}+n_{23})(n_{13}+n_{23}) \left(\frac{n_{13}n_{23}}{|p_1|^2|p_2|^2} + \frac{n_{12}n_{23}}{|p_1|^2|p_3|^2} + \frac{n_{12}n_{13}}{|p_2|^2|p_3|^2} \right). \end{aligned} \tag{3.14}$$

In the case of bubbles on two lines

$$\langle W(p_1)W^\dagger(p_2)W(p_3)\rangle_{\mathcal{PP}} = \delta^{(2)}(p_1+p_2+p_3) \sum_{n_{12},n_{23},n_{13}=1} \sum_{q,r=1} \frac{(-1)^{n_{13}}}{\pi^2 N} (-1)^{q+r}\, S_q S_r$$

$$\times \tau_1^{n_{12}+n_{13}}\, \tau_2^{n_{12}+n_{23}}\, \tau_3^{n_{13}+n_{23}} \left[\frac{\tau_1^{2q}\, \tau_2^{2r}}{(n_{12}+n_{13}+2q-1)!(n_{12}+n_{23}+2r-1)!(n_{13}+n_{23})!} \right.$$

$$+ \frac{\tau_1^{2q}\, \tau_3^{2r}}{(n_{12}+n_{13}+2q-1)!(n_{12}+n_{23})!(n_{13}+n_{23}+2r-1)!}$$

$$\left. + \frac{\tau_2^{2q}\, \tau_3^{2r}}{(n_{12}+n_{13})!(n_{12}+n_{23}+2q-1)!(n_{13}+n_{23}+2r-1)!} \right]$$

$$\times (n_{12}+n_{13})(n_{12}+n_{23})(n_{13}+n_{23}) \left(\frac{n_{13}n_{23}}{|p_1|^2|p_2|^2} + \frac{n_{12}n_{23}}{|p_1|^2|p_3|^2} + \frac{n_{12}n_{13}}{|p_2|^2|p_3|^2} \right), \qquad (3.15)$$

and finally, when only one line is affected,

$$W(p_1)W^\dagger(p_2)W(p_3)\rangle_{\mathcal{P}} = \delta^{(2)}(p_1+p_2+p_3) \sum_{n_{12},n_{23},n_{13}=1} \sum_{q=1} \frac{(-1)^{n_{13}}}{\pi^2 N} (-1)^q\, S_q \times$$

$$\times \tau_1^{n_{12}+n_{13}} \tau_2^{n_{12}+n_{23}} \tau_3^{n_{13}+n_{23}} \left[\frac{\tau_1^{2q}}{(n_{12}+n_{13}+2q-1)!(n_{12}+n_{23})!(n_{13}+n_{23})!} \right.$$

$$+ \frac{\tau_2^{2q}}{(n_{12}+n_{13})!(n_{12}+n_{23}+2q-1)!(n_{13}+n_{23})!}$$

$$\left. + \frac{\tau_3^{2q}}{(n_{12}+n_{13})!(n_{12}+n_{23})!(n_{13}+n_{23}+2q-1)!} \right]$$

$$\times (n_{12}+n_{13})(n_{12}+n_{23})(n_{13}+n_{23}) \left(\frac{n_{13}n_{23}}{|p_1|^2|p_2|^2} + \frac{n_{12}n_{23}}{|p_1|^2|p_3|^2} + \frac{n_{12}n_{13}}{|p_2|^2|p_3|^2} \right). \qquad (3.16)$$

4. Asymptotic behaviour

Since we are interested in large values of the variables τ_i, the series above have to be explicitly resummed. This is most easily done by recalling the inverse Laplace transform

$$\frac{\tau^n}{n!} = \int_{c-i\infty}^{c+i\infty} \frac{dw}{2\pi i} w^{-n-1} e^{w\tau}, \qquad c > 0\,.$$

The typical sums in eq. (3.12) are

$$\mathcal{S} \equiv \sum_{n_{12},n_{23},n_{13}=1} (-1)^{n_{13}} \frac{\tau_1^{n_{12}+n_{13}-1}\, \tau_2^{n_{12}+n_{23}-1}\, \tau_3^{n_{13}+n_{23}-1}}{(n_{12}+n_{13}-1)!(n_{13}+n_{23}-1)!(n_{12}+n_{23}-1)!}$$

$$= -\int_{c-i\infty}^{c+i\infty} \frac{dw_1 dw_2 dw_3}{(2\pi i)^3} \frac{e^{w_1\tau_1+w_2\tau_2+w_3\tau_3}}{(w_1w_2-1)(w_2w_3-1)(w_3w_1+1)}, \qquad c > 1\,. \qquad (4.1)$$

A careful treatment of those integrals leads to the asymptotic estimate

$$\mathcal{S} \simeq -\exp\left(2\sqrt{\tau_2(\tau_1+\tau_3)}\right), \qquad (4.2)$$

apart from slowly increasing power factors. A detailed calculation is reported in appendix B.

Then the extra n_{ij} factors in eq. (3.12) can be accounted for by suitable differentiations; for instance the factor n_{12} can be obtained taking derivatives with respect to the variable $\xi_{12} \equiv \tau_1\tau_2$. A little thought is enough to conclude that the leading term occurs when the derivatives act on the exponential factor, so that one can differentiate directly the asymptotic estimate and only the power factor in eq. (4.2) is changed. Actually, owing to the alternating sign with n_{13} in eq. (3.12), the other two terms, namely the ones involving n_{13}, turn out to be subleading.

Taking the relation eq. (2.12) into account and remembering that the two-line correlator $\langle W(p)W^\dagger(p)\rangle$ increases like $\exp(2\tau)$, one concludes that the *normalized* three-line correlator goes to a constant at ∞ (always disregarding power corrections).

Two remarkable features are to be noticed.

First, the increase of the correlator with increasing lengths of the lines, in spite of the fact that an interference occurs in the exchanges between $W(p_1)$ and $W(p_3)$. This effect is overwhelmed by the coherent exchanges between $W(p_1)$ and $W^\dagger(p_2)$, $W^\dagger(p_2)$ and $W(p_3)$. The increase is therefore a consequence of the parallelism of the lines, as correctly suggested in [4].

The second effect, which has not been noticed in previous treatments, is the extra $1/N$ factor; in the case of a ν-line correlator this factor would be $N^{2-\nu}$. As a consequence, multiple line correlators get more and more depressed in the 't Hooft limit.

In the case of lines with bubbles (see e.g. eq. (3.14)), the typical sums in eq. (4.1) are thereby to be replaced with

$$\mathcal{S} \equiv \sum_{n_{12},n_{23},n_{13}=1} \sum_{q_1,q_2,q_3=1} \frac{(-1)^{n_{13}}(-4)^{q_1+q_2+q_3}}{\pi\sqrt{\pi}} \frac{\Gamma(q_1+1/2)}{(q_1+1)!} \frac{\Gamma(q_2+1/2)}{(q_2+1)!} \frac{\Gamma(q_3+1/2)}{(q_3+1)!} \times \\ \times \frac{\tau_1^{n_{12}+n_{13}+2q_1-1}\,\tau_2^{n_{12}+n_{23}+2q_2-1}\,\tau_3^{n_{13}+n_{23}+2q_3-1}}{(n_{12}+n_{13}+2q_1-1)!(n_{13}+n_{23}+2q_2-1)!(n_{12}+n_{23}+2q_3-1)!}. \tag{4.3}$$

Introducing again inverse Laplace transforms we get

$$\begin{aligned} \mathcal{S} &= \sum_{n_{12},n_{23},n_{13}=1} \sum_{q_1,q_2,q_3=1} \frac{(-1)^{n_{13}}(-4)^{q_1+q_2+q_3}}{\pi\sqrt{\pi}} \frac{\Gamma(q_1+1/2)}{(q_1+1)!} \frac{\Gamma(q_2+1/2)}{(q_2+1)!} \frac{\Gamma(q_3+1/2)}{(q_3+1)!} \times \\ &\quad \times \int_{c-i\infty}^{c+i\infty} dw_1 dw_2 dw_3 e^{w_1\tau_1+w_2\tau_2+w_3\tau_3} w_1^{-(n_{12}+n_{13}+2q_1)} w_2^{-(n_{23}+n_{12}+2q_2)} w_3^{-(n_{13}+n_{23}+2q_3)} \\ &= -\sum_{q_1,q_2,q_3=1} \frac{(-4)^{q_1+q_2+q_3}}{\pi\sqrt{\pi}} \frac{\Gamma(q_1+1/2)}{(q_1+1)!} \frac{\Gamma(q_2+1/2)}{(q_2+1)!} \frac{\Gamma(q_3+1/2)}{(q_3+1)!} \times \\ &\quad \times \int_{c-i\infty}^{c+i\infty} \frac{dw_1 dw_2 dw_3}{(2\pi i)^3} \frac{e^{w_1\tau_1+w_2\tau_2+w_3\tau_3}}{(w_1w_2-1)(w_2w_3-1)(w_3w_1+1)} w_1^{-2q_1} w_2^{-2q_2} w_3^{-2q_3}, \end{aligned} \tag{4.4}$$

with $c > 1$.

The extra sums

$$\sum_{q=1} \frac{\Gamma(q+1/2)}{\Gamma(q+2)} \frac{(-4w^{-2})^q}{\sqrt{\pi}} = \frac{2w}{w+\sqrt{w^2+4}} - 1 \equiv \phi(w) \tag{4.5}$$

do not produce any singularity for $\mathrm{Re}\, w > 0$. As a consequence they are ineffective on the asymptotic behaviour at large τ_i.

JHEP06(2003)051

Coming back to eqs. (3.14), (3.15) and (3.16), additional factors of the kind $n_{ij}n_{kl}$ can be accounted for by suitable differentiations, while factors like $n_{ij} + n_{kl}$ can be cancelled, for instance, by means of a shift in the denominators, *viz*

$$\frac{n_{12}+n_{13}}{(n_{12}+n_{13}+2q_1-1)!} = \frac{1}{(n_{12}+n_{13}+2q_1-2)!} - \frac{2q_1-1}{(n_{12}+n_{13}+2q_1-1)!}\,. \tag{4.6}$$

Both procedures entail the appearance of extra powers of q_i in the numerator, which however are ineffective on the analytic behaviour of eq. (4.5), owing to the alternating sign of the sums over q_i. Considering for instance the equation above, the net result is an increase of powers in τ_i coming from the former addendum, whereas the latter can be disregarded, remaining subleading.

After a careful examination of all the terms coming from such a proliferation, it turns out (see appendix B) that the largest contribution at large N and τ_i comes from diagrams with bubbles on all the three lines and $n_{13} = 0$. It increases (in absolute value) like

$$\langle W(p_1)W^\dagger(p_2)W(p_3)\rangle \simeq -\frac{\delta^2(p_1+p_2+p_3)}{|p_1|^2|p_3|^2}\tau^{15/2}\exp(2\tau)\,, \tag{4.7}$$

where we have set $\tau \equiv \tau_2 = \tau_1 + \tau_3$, according to eq. (2.12), and chosen, for the sake of simplicity, $\tau_1 = \tau_3$.

If we now remember that the two-line correlator increases with τ like [5][4]

$$\langle W(p)W^\dagger(p)\rangle \simeq \tau^{7/2}\exp(2\tau), \tag{4.8}$$

we can easily realize that the *normalized* three-line correlator, in the case of parallel lines, increases (in absolute value) mildly with τ, namely as $\tau^{9/4}$, for fixed values of the momenta.

We notice that, although eqs. (4.7) and (4.8) followed from a perturbative analysis, having resummed all orders, they hold also at large g^2N (g fixed). This fact protects the normalized three-line correlator against suppression at large N.

Unfortunately, if the lines are not parallel, the considerations concerning colour structure can be repeated, but the result eq. (3.10) exhibits an explicit dependence on the variables a_i, b_j and c_k. In addition such a dependence is different for different graphs and this eventually prevents us from performing integrations over the line variables at a generic perturbative order (the analog of eq. (3.11)).

Owing to this limitation, we are unable to draw any concrete prediction concerning non-parallel lines, while expecting, on a purely intuitive basis, a loss of coherence and thereby a decrease with the line lengths.

5. Conclusions

In this paper a perturbative calculation of the correlator of three parallel open Wilson lines in two non-commutative space-time dimensions was performed. In the large-N planar limit, the perturbative series was fully resummed. Remarkably, θ-dependence turns out to

[4]The infinite normalization $\delta(0)$ coming from momentum conservation will here be dropped.

be trivial, intervening only through the length of the lines, just in those diagrams which are dominant at large N. As pointed out in [5], this feature is peculiar of two dimensions, where transverse degrees of freedom are absent and the theory exhibits invariance under area-preserving diffeomorphisms, and in fact the same result was previously found for the correlator of two parallel Wilson lines.

Being interested in the large argument behaviour of the three-line correlator, we were able to give an asymptotic estimate, showing an exponential growth of the correlator with the lengths of the lines, in spite of an interference effect between lines with the same orientation. This result generalizes a similar increase occurring in the two-line correlator, so that we could conclude that the *normalized* three-line correlator is still increasing like a (small) power of its argument.

A novel feature of our treatment is the appearance of a damping factor $1/N$, which implies that the correlator considered is depressed in the 't Hooft limit, at odds with the two-line analog. As the number of lines involved is increased, multiple line correlators are expected to get more and more depressed.

Actually, our treatment can, in principle, be extended to multiple (parallel) line correlators. The first novelty concerns the colour structure. A simple thought is enough to conclude that the colour pattern of eq. (3.1) is to be generalized to *cyclic colour strings*, namely strings with lines exchanging propagators only between nearest neighbours. As an example with ν lines, we consider the pattern

$$\begin{aligned}&(a_1 \dots a_{n_{\nu 1}} b_1 \dots b_{n_{12}})(b_{n_{12}} \dots b_1 c_1 \dots c_{n_{23}})\times\\ &\times(c_{n_{23}} \dots c_1 d_1 \dots d_{n_{34}}) \dots (z_1 \dots z_{n_{\nu-1\nu}} a_{n_{\nu 1}} \dots a_1) = N^{2-\nu}\left(\frac{N}{2}\right)^n,\end{aligned} \tag{5.1}$$

n representing, as usual, the total number of propagators. Notice the power $N^{2-\nu}$, which shows how *connected* configurations are subleading with respect to disconnected ones. Indeed, the latter can be realized as a product of the former. For the sake of clarity, for the product of two connected correlators with ν_1, ν_2 lines, respectively, at the perturbative order $n = n_1 + n_2$, from eq. (5.1) we can infer the behaviour

$$N^{2-\nu_1}\left(\frac{N}{2}\right)^{n_1} N^{2-\nu_2}\left(\frac{N}{2}\right)^{n_2} = N^{4-\nu}\left(\frac{N}{2}\right)^n, \qquad \nu = \nu_1 + \nu_2\,. \tag{5.2}$$

Cyclic configurations as in eq. (5.1) are colour-leading, but are not the only ones. For instance, with $\nu = 4$, the configuration

$$\begin{aligned}&(a_1 \dots a_{n41} f_1 \dots f_{n_{13}} b_1 \dots b_{n_{12}})(b_{n_{12}} \dots b_1 c_1 \dots c_{n_{23}})\times\\ &\times(c_{n_{23}} \dots c_1 f_{n_{13}} \dots f_1 d_1 \dots d_{n_{34}})(d_{n_{34}} \dots d_1 a_{n_{41}} \dots a_1) = N^{-2}\left(\frac{N}{2}\right)^n,\end{aligned} \tag{5.3}$$

is still leading, whereas

$$\begin{aligned}&(a_1 \dots a_{n41} f_1 \dots f_{n_{13}} b_1 \dots b_{n_{12}})(b_{n_{12}} \dots b_1 g_1 \dots g_{n_{24}} c_1 \dots c_{n_{23}})\times\\ &\times(c_{n_{23}} \dots c_1 f_{n_{13}} \dots f_1 d_1 \dots d_{n_{34}})(d_{n_{34}} \dots d_1 g_{n_{24}} \dots g_1 a_{n_{41}} \dots a_1) = N^{-4}\left(\frac{N}{2}\right)^n\end{aligned} \tag{5.4}$$

is not. For $\nu = 4$ the colour leading configurations can be counted; then the "geometric" structure can be worked out, suitably generalizing the treatment leading to eq. (3.4). After a long (and cumbersome) calculation, indications emerge of a coherent increase with the line lengths of the kind already described.

Beyond $\nu = 4$, both the singling out of the colour leading configurations and the subsequent evaluation of the geometrical factors become almost intractable. On the other hand, in our opinion, such an effort would not be worthwhile since it could hardly add new understanding from the point of view of the physics involved.

A different consideration might be deserved by the possibility of performing non-perturbative evaluations based on a compaction of the theory on a torus, followed by a Morita mapping, repeating the treatment developed in [5] for the two-line correlator. Here the difficulty concerns the harmonic analysis on the commutative torus when three (or more) cycles are involved and the subsequent identification of suitable saddle points (if any), in order to concretely evaluate the large N, large τ_i limit. It would be nice to discover a coherent increase of multiple line correlators in a non-perturbative context.

These and related developments will be deferred to future investigations and the results will be reported elsewhere.

Acknowledgments

Discussions with A. Torrielli are gratefully acknowledged. One of us (F.V.) wishes to thank the CERN-TH Division for hospitality during the completion of this work.

A. Details of calculations appearing in section 3

Starting from eq. (3.4), we notice that the polynomial $\mathcal{P}$ in the Moyal phase $e^{i\mathcal{P}(P_i,Q_j,R_k;\theta,\mathbf{p})}$ is (for the relevant class of graphs) at most linear in P_i, Q_j, R_k and, since $p_r\theta p_s \equiv 0$, the constant term is missing. We can then factorize $e^{i\mathcal{P}}$, provided we replace $\xi_i - \zeta_i, \eta_j - \xi'_j, \eta'_k - \zeta'_k$ with a_i, b_j, c_k defined as in eq. (3.6). Furthermore, we introduce three complex variables U, V, W in order to express the δ-functions enforcing momentum conservation as integrals, e.g.

$$\delta^{(2)}\left(\sum_i P_i + \sum_k R_k - p_1\right) = (2\pi)^{-2} \int d^2U e^{i(\sum_i P_i + \sum_k R_k - p_1)\cdot U} . \tag{A.1}$$

We end up with an expression of the form

$$(2\pi)^{-6} \int e^{-i(p_1\cdot U + p_2\cdot V + p_3\cdot W)} f(U-V, V-W, W-U) d^2U d^2V d^2W ; \tag{A.2}$$

it can be rewritten as

$$(2\pi)^{-6} \int e^{-\frac{i}{3}(p_1+p_2+p_3)\cdot(U+V+W)} \Big[e^{-\frac{i}{3}((p_1-p_2)\cdot(U-V)+(p_2-p_3)\cdot(V-W)+(p_3-p_1)\cdot(W-U))} \times \\ \times f(U-V, V-W, W-U) \Big] d^2U d^2V d^2W , \tag{A.3}$$

from which the expected momentum conserving δ-function factorizes

$$(2\pi)^{-4}\delta^{(2)}(p_1+p_2+p_3)\int\left[e^{-\frac{i}{3}((p_1-p_2)\cdot x+(p_2-p_3)\cdot y+(p_3-p_1)\cdot z)}f(x,y,z)\right]\times \\ \times\,\delta^{(2)}(x+y+z)\,d^2x\,d^2y\,d^2z\,. \tag{A.4}$$

Now we introduce a two-component variable P to exponentiate $\delta^{(2)}(x+y+z)$, as in eq. (A.1), and then perform the integrations over P_i, Q_j, R_k. By recalling eq. (2.11), we finally obtain the factorized form eq. (3.6).

Next, we have to deal with three integrals of the form

$$I(P,\tilde{p}_1,\mathbf{a})\equiv\int d^2x\,e^{i(P-\tilde{p}_1)\cdot x}\prod_{i=1}^{n_{12}}\frac{x-a_i}{(x-a_i)^*} \tag{A.5}$$

(see eq. (3.7)). It is understood here that we must perform a symmetric integration around the poles $x=a_i$; hence we need to decompose the integrand in eq. (A.5) in simple fractions. We can use almost verbatim a result from [5]. The denominator decomposes as in eq. (8.2) therein, apart from minor variable redefinitions

$$\prod_{i=1}^{n_{12}}\frac{1}{(x-a_i)^*}=\sum_{i=1}^{n_{12}}\frac{1}{(x-a_i)^*}\prod_{j\neq i}\frac{1}{(a_i-a_j)^*}\,. \tag{A.6}$$

In each term we shift the integration variable as $x-a_i\to x$, expand the polynomial in the numerator in the new variable x, perform a symmetric integration according to

$$\int d^2x\,e^{-iP\cdot x}\,\frac{x^m}{x_-}=\frac{\pi(-2i)^{m+1}m!}{P_-^{m+1}} \tag{A.7}$$

and obtain eq. (3.7).[5]

Let us now consider the integral

$$\int d^2P\,\frac{e^{iP\cdot(a_i+b_j+c_k)}}{(P-\tilde{p}_1)_-^{m_1+1}(P-\tilde{p}_2)_-^{m_2+1}(P-\tilde{p}_3)_-^{m_3+1}}\,, \tag{A.8}$$

which has to be computed via symmetric integration around the singularities of the integrand. Again, we need to decompose the integrand in simple fractions, which can subsequently be integrated by means of standard techniques. Hence we must find suitable coefficients c_{r,n_r} so as to match

$$\frac{1}{(P-\tilde{p}_1)_-^{m_1+1}(P-\tilde{p}_2)_-^{m_2+1}(P-\tilde{p}_3)_-^{m_3+1}}=\sum_{r=1}^{3}\sum_{n_r=1}^{m_r+1}c_{r,n_r}\frac{1}{(P-\tilde{p}_r)_-^{n_r}}\,. \tag{A.9}$$

The coefficients c_{r,n_r} can be obtained by imposing that both sides of eq. (A.9), when multiplied by an arbitrary power of $(P_- - \tilde{p}_{i-})$, have coincident residues at each pole $P_- = \tilde{p}_{i-}$. The result is

$$c_{r,n_r}=\sum_{\{l_s\,:\,l=m_r-n_r\}}(-1)^l\prod_{s\neq r}\binom{m_s+l_s}{l_s}\frac{1}{(\tilde{p}_r-\tilde{p}_s)_-^{m_s+l_s+1}}\,, \tag{A.10}$$

[5]The integrals in eqs. (A.5) and (A.7) are to be intended in the sense of the theory of distributions.

where $l \equiv \sum_{s\neq r} l_s$. The single terms in the decomposition of eq. (A.9) can then be symmetrically integrated around their pole, the result being

$$\int d^2P \frac{e^{iP\cdot(a_i+b_j+c_k)}}{(P-\tilde{p}_r)_-^{n_r}} = e^{i\tilde{p}_r\cdot(a_i+b_j+c_k)} \frac{\pi i^{n_r} 2^{-n_r+2}}{(n_r-1)!} \frac{(a_i+b_j+c_k)^{n_r-1}}{(a_i+b_j+c_k)^*}\,, \tag{A.11}$$

whence eq. (3.10) follows.

We can now single out in eq. (3.10) the only relevant terms. Since all the parameters have the same complex phase (owing to the parallelism of the three Wilson lines), it is easily seen that the phase of each summand in eq. (3.10) depends only on n_{12}, n_{13}, n_{23}, which are fixed for a given graph, but not on any of the indices which are summed over. After factorizing a common phase,[6] we can replace a_i^*, b_j^*, c_k^* with a_i, b_j, c_k, respectively, and $\tilde{p}_i$ with $|p_i|$ in eqs. (3.8) and (3.10). Hence, on one hand eq. (3.10) is symmetric under the exchange of any pair of a_i, or b_j, or c_k, on the other hand, the common denominator in the sum over i, j, k, being the product of the three Vandermonde determinants built with a_i, b_j, c_k, is antisymmetric. It follows that the numerator of the sum must be antisymmetric too, and this entails (by repeated use of Ruffini's theorem) that it has those determinants as factors and therefore the sum must have non-negative degree in the parameters. Since every term has manifestly non-positive degree, it follows that only zero degree terms survive. One can realize that those originate e.g. from the first term of the r.h.s. of eq. (3.9) when the conditions $m_r = 1$, $l_r = 0$, $r = 2, 3$ are fulfilled, and from the other two terms when analogous conditions hold.

They can be evaluated as follows. Let us consider for the sake of example again the first contribution to the function $\mathcal{K}$ appearing in eq. (3.9). When inserted in eq. (3.10), two out of three functions V are forced to equal unity, namely those depending on $\mathbf{b}$ and $\mathbf{c}$, since $m_2 = m_3 = 1$. The relevant sum (i.e. apart from factors that are independent of the indices summed over) reduces to

$$\sum_{m_1=1}^{n_{12}} (-1)^{m_1} \sum_{i=1}^{n_{12}} \sum_{j=1}^{n_{13}} \sum_{k=1}^{n_{23}} \sum_{j_1<j_2<\ldots<j_{n_{12}-m_1}} \frac{(a_i-a_{j_1})\ldots(a_i-a_{j_{n_{12}-m_1}})}{\prod_{j\neq i}(a_i-a_j)} (a_i+b_j+c_k)^{m_1-1}\,. \tag{A.12}$$

When we sum over i while keeping j, k and m_1 fixed, the result must be symmetric under the exchange of any pair of a_i; hence (by the same token as above) only zero degree terms survive, and we can set b_j, c_k equal to zero, as they multiply negative powers of a_i. The sum over n_{13} and n_{23} then merely results in a multiplicity factor $n_{13}n_{23}$.

Now in the equivalent expression

$$\sum_{i=1}^{n_{12}} \sum_{j_1<j_2<\ldots<j_{m_1-1}} \frac{a_i^{m_1-1}}{(a_i-a_{j_1})\ldots(a_i-a_{j_{m_1-1}})} \tag{A.13}$$

we can see that each term depends on exactly m_1 out of the n_{12} parameters a_i.

There are just $\binom{n_{12}}{m_1}$ such sets, and to each of them again the same argument applies, allowing only zero degree terms to survive. These are pure numbers, and can actually

[6]Such a phase will eventually cancel against an opposite one originating from the integration over the line variables [5].

be seen to equal unity (by means of counting the occurrences of a given monomial in the numerator and in the denominator). Finally, eq. (A.12) becomes

$$\sum_{m_1=1}^{n_{12}} \binom{n_{12}}{m_1} (-1)^{m_1} n_{13} n_{23} = -n_{13} n_{23} \,. \tag{A.14}$$

Analogous results are found when the other two terms contributing to the function $\mathcal{K}$ in eq. (3.9) are considered, so that we can conclude eq. (3.11) follows.

B. Asymptotic estimates

We can infer the asymptotic behaviour of the various contributions eqs. (3.12), (3.14) – (3.16) for large τ_i from the Laplace transform representation. In what follows, it is understood that the variables τ_i are sent to infinity, while keeping their ratios finite (*e.g.* $\tau_1 = \tau_3 = \frac{\tau_2}{2}$). We proceed to estimate the inverse Laplace transform by looking at each stage for the rightmost singularity, which will give the exponentially dominant contribution. Since we are not considering disconnected graphs, we have to deal separately with contributions where $n_{12}, n_{13}, n_{23} \neq 0$, and with those where one (and one only) among n_{12}, n_{13}, n_{23} vanishes. Besides, for each case, there are contributions arising from graphs without self energy bubbles, and with bubbles on one, two, or all the three Wilson lines. Here we will sketch how the asymptotic estimate is achieved in the simplest cases (no bubbles), and in the dominant case, which turns out to consist of graphs with bubbles on three lines, and either $n_{12}, n_{13}, n_{23} \neq 0$ or only $n_{13} = 0$.

Let us start from the case without bubbles. In eq. (4.1) we can integrate first over w_1, the sign of the exponential allowing us to shift the contour to the left; the singularities are in $w_1 = \frac{1}{w_2}$ and $w_1 = -\frac{1}{w_3}$. Since w_2, w_3 (as well as w_1) vary along a vertical line with a real part $c > 1$, the latter pole has negative real part and produces a negligible (exponentially suppressed) contribution when compared to the former; we will henceforth drop this and similar terms altogether. Hence

$$\mathcal{S} \simeq -\int_{c-i\infty}^{c+i\infty} \frac{dw_2 dw_3}{(2\pi i)^2} \frac{e^{\frac{\tau_1}{w_2} + w_2\tau_2 + w_3\tau_3}}{(w_2 + w_3)(w_3 w_2 + 1)} \,, \qquad c > 1 \,. \tag{B.1}$$

Next we integrate over w_3, and drop the pole in $w_3 = -w_2$

$$\mathcal{S} \simeq -\int_{c-i\infty}^{c+i\infty} \frac{dw_2}{(2\pi i)} \frac{e^{\frac{(\tau_1+\tau_3)}{w_2} + w_2\tau_2}}{(w_2^2 + 1)} \,, \qquad c > 1 \,. \tag{B.2}$$

The integral in eq. (B.2) can be approximated through the saddle-point method: one finds the stationary point of the quadratic form in the exponent, which turns out to be $w_2 = \frac{\tau_1+\tau_3}{\tau_2}$, then translates the contour to $c = \frac{\tau_1+\tau_3}{\tau_2}$, and wisely expands the integrand about the stationary point (that is, both the factor multiplying the exponential and the *exponent* are Taylor expanded, the quadratic part in the exponent gives rise to a gaussian integration, higher terms are Taylor expanded if needed). In this way we get

$$\mathcal{S} \simeq -i\sqrt{\pi} \frac{(\tau_2(\tau_1+\tau_3))^{\frac{1}{4}}}{\tau_2} e^{2(\tau_2(\tau_1+\tau_3))^{\frac{1}{2}}} \left(\frac{1}{\frac{\tau_1+\tau_3}{\tau_2} + 1} + \ldots \right) \,, \tag{B.3}$$

where the neglected terms are suppressed by at least a factor of $1/\tau_i$. It is very useful to notice the following point: the asymptotics of eq. (3.12) contains *derivatives* of $\mathcal{S}$ with respect to τ_i, evaluated on the constraint $\tau_1 + \tau_3 \equiv \tau_2$. Indeed the $n_{12}n_{13}$, ... factors can be represented introducing new variables $\xi_{ij} \equiv \tau_i\tau_j\,, i < j$, as $n_{ij} \to \xi_{ij}\partial_{\xi_{ij}}$.

We should first differentiate and then impose the constraint. Nevertheless, in the spirit of this approximation, it is easy to realize that when any number of derivatives act on anything but $e^{2(\tau_2(\tau_1+\tau_3))^{\frac{1}{2}}}$ in $\mathcal{S}$, they produce terms which are power suppressed in τ_i. Hence we are free to impose the constraint on everything but the exponential, and differentiate the exponential alone when evaluating the asymptotics of eq. (3.12). With this convention in mind, we can write as well

$$\mathcal{S} \simeq -i\frac{\sqrt{\pi}}{2\sqrt{\tau_2}}e^{2(\tau_2(\tau_1+\tau_3))^{\frac{1}{2}}} + \dots . \tag{B.4}$$

Then, keeping in mind that derivatives are relevant when acting on the exponential, we see that only the term $\frac{n_{12}n_{23}}{p_{1-}^2 p_{3-}^2}$ contributes in the asymptotic limit.

When we consider graphs with $n_{13} \equiv 0$ and no bubbles, it is easy to see that replacing the sum over n_{13} with the single $n_{13} = 0$ term results in the absence of the factor $\frac{-1}{w_1 w_3+1}$ in the Laplace transform. This factor does not contribute dominant poles in the Laplace inversion and only amounts to a constant when evaluated on the relevant singularities. Then in eq. (B.2) the factor $\frac{-1}{w_2^2+1}$ is replaced with $\frac{1}{w_2^2}$, and, since eventually the constraint $w_2 = \frac{\tau_1+\tau_3}{\tau_2} = 1$ is imposed, the term with $n_{13} = 0$ is (in the large τ limit) minus twice the term with $n_{12}, n_{13}, n_{23} \neq 0$. Terms with $n_{12} = 0$ or $n_{23} = 0$ are exponentially suppressed, since one of the leading singularities is missing.

When we come to contributions with bubbles on one or more lines, the asymptotic estimate is very similar, though more involved. We consider the example with $n_{12}, n_{13}, n_{23} \neq 0$ and bubbles on all the lines, all the other cases behaving analogously.

First, when comparing eq. (3.14) with eq. (3.12), we notice the following differences:

- denominators like $\frac{1}{(n_{12}+n_{13}-1)!}$, ... are replaced with $\frac{n_{12}+n_{13}}{(n_{12}+n_{13}+2q_1-1)!}$, ...
- the power of τ_i is increased by $2q_i$
- the factor $(-1)^{(q_1+q_2+q_3)}S_{q_1}S_{q_2}S_{q_3}$ appears.

Due to the modified powers of τ_i, factors like n_{12}, ... cannot be exactly reproduced by derivatives with respect to τ_i, but the mismatch is subleading. Indeed, replacing e.g. n_{12} with $\xi_{12}\partial_{\xi_{12}}$ means to approximate $n_{12} \simeq n_{12} + (q_1 + q_2 - q_3)$, and the difference is subleading because, as we shall see, q_i factors can only affect the constant multiplying the leading exponential behaviour, while the derivatives $\xi_{ij}\partial_{\xi_{ij}}$ introduce powers of ξ_{ij}. With this in mind, we can recover the various factors n_{ij} by differentiation, factor out $\tau_1\tau_2\tau_3$ as we did for eq. (3.12), and then Laplace transform. The sums over q_i as in eq. (4.5) have no pole. Multiplying S_{q_i} by q_i amounts to act on $\phi(w)$ with $-\frac{1}{2}w\frac{d}{dw}$, and again no poles are produced (hence justifying our former claim). The presence of the factors $\phi(w_i)$ boils down, in the saddle point approximation, to powers of $\phi(1) = \frac{\sqrt{5}-3}{2}$, while the extra derivatives contribute powers of τ_i. The terms with more derivatives, i.e. more factors of n_{ij}, dominate; they come from bubbles on the three lines, both with $n_{ij} \neq 0$ and with $n_{13} = 0$, the latter being, by the same argument discussed above, twice the former and opposite in sign.

References

[1] L. Alvarez-Gaumé and S.R. Wadia, *Gauge theory on a quantum phase space*, *Phys. Lett.* **B 501** (2001) 319 [hep-th/0006219];
S.-J. Rey and R. von Unge, *S-duality, noncritical open string and noncommutative gauge theory*, *Phys. Lett.* **B 499** (2001) 215 [hep-th/0007089];
S.R. Das and S.-J. Rey, *Open wilson lines in noncommutative gauge theory and tomography of holographic dual supergravity*, *Nucl. Phys.* **B 590** (2000) 453 [hep-th/0008042];
J.A. Harvey, *Topology of the gauge group in noncommutative gauge theory*, hep-th/0105242.

[2] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048];
R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[3] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, *Nucl. Phys.* **B 573** (2000) 573 [hep-th/9910004].

[4] D.J. Gross, A. Hashimoto and N. Itzhaki, *Observables of non-commutative gauge theories*, *Adv. Theor. Math. Phys.* **4** (2000) 893 [hep-th/0008075].

[5] A. Bassetto and F. Vian, *Wilson line correlators in two-dimensional noncommutative Yang-Mills theory*, *J. High Energy Phys.* **10** (2002) 004 [hep-th/0207222].

[6] G. Mandal, S.-J. Rey and S.R. Wadia, *Quantum aspects of GMS solutions of noncommutative field theory and large-N limit of matrix models*, *Eur. Phys. J.* **C 24** (2002) 495 [hep-th/0111059].

[7] L.D. Paniak and R.J. Szabo, *Open Wilson lines and group theory of noncommutative Yang-Mills theory in two dimensions*, *J. High Energy Phys.* **05** (2003) 029 [hep-th/0302162].

[8] T.T. Wu, *Two-dimensional Yang-Mills theory in the leading* $1/N$ *expansion*, *Phys. Lett.* **B 71** (1977) 142.

[9] S. Mandelstam, *Light cone superspace and the ultraviolet finiteness of the* $N = 4$ *model*, *Nucl. Phys.* **B 213** (1983) 149;
G. Leibbrandt, *The light cone gauge in Yang-Mills theory*, *Phys. Rev.* **D 29** (1984) 1699.

[10] A. Bassetto, G. Nardelli and R. Soldati, *Yang-Mills theories in algebraic noncovariant gauges: canonical quantization and renormalization*, World Scientific, Singapore 1991.

[11] A. Bassetto, M. Dalbosco, I. Lazzizzera and R. Soldati, *Yang-Mills theories in the light cone gauge*, *Phys. Rev.* **D 31** (1985) 2012.

[12] A.K. Das and M.M. Sheikh-Jabbari, *Absence of higher order corrections to noncommutative Chern-Simons coupling*, *J. High Energy Phys.* **06** (2001) 028 [hep-th/0103139].

[13] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *J. High Energy Phys.* **02** (2000) 020 [hep-th/9912072].